

University of California at Santa Barbara

Physics Department

Physics 225A Homework 3

Problems 2,4, and 5 are even more optional than the other.

Do them if you feel like it.

Even if you do not do them, you might want to note the following interesting results:

- (a) The irreducible representations of abelian groups are one dimensional (Problem 2).
- (b) For any Lie Group, the commutator of two generators can be written as a sum, with appropriate coefficients, of all the generators (Problem 4).
- (c) For a Lie Group it is possible to come up with an operator which commutes with all the generators (Casimir operator). This operator is the sum, with appropriate coefficients, of the product of pairs of generators (Problem 5).

• Problem 1

From the definition of a group given in class, show that for a group G

- (a) the identity element is unique
- (b) every $a \in G$ has a unique inverse
- (c) $\forall a \in G, (a^{-1})^{-1} = a$
- (d) $\forall a, b \in G, (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$

• Problem 2

Consider an irreducible representation of a group in a space L . Let A be a fixed operator in L , and let $G(a)$ be the operator in this vector space corresponding to the group element a . Show that if $A \cdot G(a) = G(a) \cdot A$ for all a , then $A = \lambda I$, where λ is a number and I is the identity operator (Schur's first lemma).

If you cannot do this yourself, you might want to look up the proof of Schur's first lemma in a Group Theory textbook.

Use this result to show that the irreducible representations of an abelian group are one-dimensional. (An abelian group is a group for which the group multiplication operation is commutative).

• **Problem 3**

Halzen and Martin, Problem 2.5. Show also that for an irreducible representation there cannot be more than one basis vector with the same eigenvalue of the third component of angular momentum.

• **Problem 4**

Prove that

$$[X_i, X_j] = \sum_k c_{ij}^k X_k$$

where the X_i are generators of a Lie group G . Use the fact that if $g(\vec{a}) \in G$ and $g(\vec{b}) \in G$ then

$$g(\vec{a}) \cdot g(\vec{b}) \cdot g(\vec{a})^{-1} \cdot g(\vec{b})^{-1} = g(\vec{c}) \in G$$

and expand to terms of order $\vec{a}\vec{a}$, $\vec{a}\vec{b}$, and $\vec{b}\vec{b}$.

• **Problem 5**

Consider the commutation relations for the generators of a Lie group :

$$[X_i, X_j] = \sum_k c_{ij}^k X_k$$

Show that the Casimir operator

$$C = \sum_{pq} (g^{-1})_{pq} X_p X_q$$

commutes with all generators. g^{-1} is the inverse of the matrix g defined as

$$g_{ij} = \sum_{ts} c_{it}^s c_{js}^t.$$

Hint: you may want to prove that the matrix

$$A_{pr} = \sum_q (g^{-1})_{pq} C_{qa}^r$$

is such that $A_{pr} = -A_{rp}$ for all a . This is most easily done by showing that $B = g \cdot A \cdot g$ is also antisymmetric. Also, you may want to make use of the Jacobi identity

$$[X_j, [X_k, X_l]] + [X_l, [X_j, X_k]] + [X_k, [X_l, X_j]] = 0$$

• **Problem 6**

The parity of the charged pion can be deduced from the capture of π^- in deuterium,

$$\pi^- d \rightarrow nn.$$

The capture takes place from an $S=0$ state. Show that this reaction implies that the pion has odd parity ($P_\pi = -1$). NB: deuterons have spin 1 and even parity.

• **Problem 7**

Consider a $\pi^0\pi^0$ system in a state of angular momentum l . This could occur in the decay $X \rightarrow \pi^0\pi^0$ if the spin of the decay particle is $S_X = l$, since the pion is spinless.

- (a) What values of l are allowed for a $\pi^0\pi^0$ system?
- (b) The $\pi^0\pi^0$ system can be described by a wave function

$$|\Psi\rangle = |\text{space}\rangle \otimes |\text{isospin}\rangle$$

What is the symmetry of the space part of the wavefunction under interchange of the two pions? What can we say about the allowed values of l ?

- (c) Is the decay $\rho^0 \rightarrow \pi^0\pi^0$ allowed? If so, by what interaction?

• **Problem 8**

In which isospin states can (a) $\pi^+\pi^-\pi^0$, (b) $\pi^0\pi^0\pi^0$ exist?

- **Problem 9**

Halzen and Martin, Problem 2.6

- **Problem 10**

Consider the operator

$$G = CR = C \exp(i\pi I_2) \quad ,$$

i.e. the operation G consists of a rotation R in isospin space about the 2-axis, followed by charge conjugation. Show that

(a) the π^0 is an eigenstate of the G operator with eigenvalue (“G-parity”) $G = -1$,

(b) the charged pions must have $G=1$ or $G = -1$.

Prove (a) and (b) without using the expressions for the π states in terms of the u and d states derived in class (or in Halzen and Martin, page 43, with $p \rightarrow u$ and $n \rightarrow d$). Just use the fact that the pions form an isospin triplet, that the π^0 must have $C=1$ (since $\pi^0 \rightarrow \gamma\gamma$), and that the π^- is the antiparticle of the π^+ .

(c) With our phase convention for the π states (Halzen and Martin, page 43), show that $G = -1$ for the charged pions also.

You may want to use results from the previous problem.